An absorber theory of acoustical radiation

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An absorber theory of acoustical radiation is developed by analogy to Wheeler–Feynman electrodynamics. Using a method due to Feynman the acoustic index of refraction for a system of soft spherical scatterers is calculated. The effects of these scatterers on an acoustic source are summed to obtain the usual radiation reaction.

Subject Classification: 11.2, 11.8.

INTRODUCTION

In their classic paper of 1945, Wheeler and Feynman[1] demonstrated that the reaction force on a radiating charge can be understood as the interaction between the charge and a universe of distant receivers. Their formulation was based on retaining the usually discarded advanced solution of the wave equation and demonstrating that the advanced fields of the receivers, radiating in response to the retarded field of the source, contribute exactly the proper field at the source to account for the radiation reaction. Their approach has the advantage that no reference is made to singular self-fields and that advanced fields are not dismissed *a priori* as being unphysical. We feel that the formal analogy between the equations of electromagnetic and acoustic radiation is so striking that it literally invites an application of the Wheeler–Feynman formulation to acoustics. The reader might wonder how the electromagnetic field, which seems to need no medium in which to propagate, and the acoustic field, which apparently demands a medium, can be so glibly placed in apposition. The answer to this question lies in the structure of the wave equation which is applicable to both:

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}=0. \tag{1}$$

In vacuum electrodynamics, ϕ is any component of the vector potential and $v^2=\epsilon_0\mu_0$. In the theory of sound propagation, ϕ is the excess pressure and $v^2=\gamma p/\rho$, where p is the equilibrium pressure, ρ is the equilibrium density, and γ is the adiabatic constant. Insofar as the form of the wave equation is concerned, a "medium" is as much in evidence in electrodynamics as it is in acoustics. While it cannot be claimed that the vacuum provides a *mechanism* for the propagation of light, it nevertheless influences that propagation through its properties ϵ_0 and μ_0 which, together, determine the propagation velocity. In acoustics, it is the medium properties p, ρ, and γ that determine the characteristic velocity of propagation for sound. We shall simply assign a passive role to the acoustic medium and shift the onus for radiative effects onto the receivers that are distributed throughout its volume. This, as we shall presently point out in greater detail, is precisely what is done by Wheeler and Feynman.

The body of this paper will be devoted to the following topics: (1) a derivation of the acoustical radiation reaction using two methods that are traditional in electrodynamics, conservation of energy and Dirac's prescription[2]; and (2) a derivation of the acoustic index of refraction for a system of soft spheres and the use of this index to obtain the radiation reaction via the Wheeler–Feynman formulation.

I. TRADITIONAL METHODS

A. Energy Balance

We consider as our source of radiation a sphere of equilibrium radius r_0 that pulsates radially with a harmonically time-dependant velocity[3]

$$V=V_0\exp(-i\omega t). \tag{2}$$

If, as usual, we discard the advanced solution of the radial wave equation, we obtain for the pressure field at large distances the expression[4]

$$p(r,t)=-(i\omega\rho r_0^2V_0/r)\exp[i\omega(r/v-t)], \tag{3}$$

where we will always assume that ω/vr_0 is very much smaller than one. Using Eq. 3, we find the total radiated power to be[5]:

$$\Pi=2\pi\omega^2\rho r_0^4V_0^2/v.$$

Since the source must be doing work to be losing energy, we may find the force of reaction by imposing the energy balance condition over a period τ:

$$\int_0^\tau \frac{2\pi\omega^2\rho r_0^4 V_0^2}{v}dt=\int_0^\tau F_{\text{reac}} V dt. \quad (5)$$

If we assume that the reaction force has a harmonic time dependence, we obtain:

$$F_{\text{reac}}=(4\pi\omega^2 r_0^3 \rho V_0/v)\exp(-i\omega t). \quad (6)$$

B. Dirac's Prescription

In 1939 Dirac[3] proposed that the correct force of radiation reaction on a radiating electron could be obtained by assuming that the electron interacts with its own field through a combination which Dirac called the radiation field:

$$E_{\text{rad}}=\tfrac{1}{2}(E_{\text{ret}}-E_{\text{adv}}). \quad (7)$$

In Eq. 7 E_{ret} is the usual retarded field of the electron and E_{adv} is its advanced field. This combination lacks the singularity that E_{ret} alone would have at the location of a point electron; moreover, the expression qE_{rad} gives the correct value for the radiation reaction on an electron.

We can show that the prescription given by Dirac also applies in the case of acoustic radiation. Defining

$$p_{\text{rad}}=\tfrac{1}{2}(p_{\text{ret}}-p_{\text{adv}}), \quad (8)$$

we should expect to find that

$$F_{\text{reac}}=4\pi r_0^2 p_{\text{rad}}(r_0). \quad (9)$$

In order to calculate $p_{\text{rad}}(r_0)$ we must use the solution of the wave equation that is valid at the surface of the source. The retarded solution can be shown to be[4]

$$p_{\text{ret}}(r,t)=-\frac{i\omega\rho r_0^2 V_0}{r(1+k^2r_0^2)^{\frac{1}{2}}}\times\exp\{-i[\omega t-k(r-r_0)+\phi_0-\pi/2]\}, \quad (10)$$

where

$$\cos\phi_0=kr_0(1+k^2r_0^2)^{-\frac{1}{2}}. \quad (11)$$

The advanced solution is generally not presented in texts, but it is easily calculated:

$$p_{\text{adv}}(r,t)=\frac{i\omega\rho r_0^2 V_0}{r(1+k^2r_0^2)^{\frac{1}{2}}}\times\exp\{-i[\omega t+k(r-r_0)-\phi_0-\pi/2]\}. \quad (12)$$

The radiation field is thus

$$p_{\text{rad}}(r,t)=-\frac{\omega\rho r_0^2 V_0}{r(1+k^2r_0^2)^{\frac{1}{2}}}\times\cos[k(r-r_0)-\phi_0]\exp(-i\omega t). \quad (13)$$

Evaluating this field at the surface of the source and calculating the resulting force gives

$$F_{\text{reac}}(r_0,t)=\frac{4\pi\omega^2\rho r_0^4 V_0}{v(1+k^2r_0^2)}\exp(-i\omega t). \quad (14)$$

Making the assumption that $k^2r_0^2$ is very much less than unity, we finally obtain the correct force of radiation reaction as in Eq. (6)[6]:

$$F_{\text{reac}}=(4\pi\omega^2\rho r_0^4 V_0/v)\exp(-i\omega t). \quad (15)$$

II. WHEELER–FEYNMAN METHOD

Wheeler and Feynman take the position that radiation is a phenomenon requiring *both* an emitter and a receiver. They substantiate this position by proving that the force of radiation reaction, the only radiative effect which requires the use of self-fields in its explanation can, in fact, be explained without resorting to the self-field of the radiating source. A brief sketch of their demonstration is as follows. First, postulate that a charge interacts only with other charges, never with itself. Thus a single charge in an otherwise empty universe cannot radiate. Second, postulate that the interaction between any pair of charges is mediated by a field that is one-half the difference between the usual retarded field and the advanced field. Third, assume that the emitter (a dipole radiator in their demonstration) is surrounded by a universe filled with receivers that are capable of reradiating. These receivers, when excited by the retarded part of the radiation field of the emitter, then produce their own retarded and advanced fields. When the index of refraction of the receiver filled universe is properly accounted for, it is shown that the advanced part of the receiver fields combine with just the proper phase differences to cancel everywhere except at the position of the emitter. At that position, the advanced receiver fields have a strength exactly equal to one-half the difference between the retarded and advanced fields of the emitter, which is precisely the field Dirac demonstrated would properly account for the force of radiation reaction. In the Wheeler–Feynman formulation, however, this field is no longer due to the self-fields of the emitter, but rather to the advanced fields of the receivers.

This all too brief sketch omits a wealth of intriguing features to be found in the original paper. We shall only concern ourselves here with a proof that the acoustic radiation reaction can also be viewed as a result of the advanced fields of receivers, summed with a proper attention given to phase differences induced by the acoustic index of refraction. The other features of the Wheeler–Feynman formulation will follow *mutis mutandis*, and they will not be set forth herein.

III. CALCULATION OF THE ACOUSTIC INDEX OF REFRACTION

Anticipating the need for the index of refraction to account for phase differences in the radiation from varying depths within the absorber, we proceed to calculate this quantity from elementary considerations by using a model due to Feynman[7] of the electromagnetic index of refraction. We picture an infinite, very thin sheet of width ΔL which contains N soft spheres per unit volume. A plane pressure wave is impinging on this sheet from the left. If we assume that the only effect of this sheet is to reduce the propagation velocity of the impinging wave by a factor n, the index of refraction, then we calculate the time delay introduced by the sheet to be

$$\Delta t=(n-1)\Delta L/v. \quad (16)$$

If the sheet were absent, the pressure at a point x to the right of where the sheet would be is $p_0\exp[-i\omega(t-x/v)]$; however, owing to the presence of the sheet, there is an additional field of acoustic reradiation. The sole effect of this additional field is to cause the actual field, p_{after}, to lag the original field (with sheet absent) by the phase angle $\omega\Delta t$. The field at point x must therefore be:

$$p_{\text{after}}=p_0\exp\{-i\omega[t-(n-1)\Delta L/v-x/v]\}. \quad (17)$$

If n is close to unity, we can make use of a power series expansion and obtain the actual field as the expected sum of an incident plus a reradiated field:

$$p_{\text{after}}(x,t)=p_0\exp[-i\omega(t-x/v)] + \frac{i\omega(n-1)\Delta L}{v}p_0\exp[-i\omega(t-x/v)]. \quad (18)$$

We will now calculate the actual reradiated field by giving detailed consideration to the effects of the spherical receivers in the sheet. By comparing this field with the expression in Eq. 18 we can extract a value for n.

Let us take as a typical soft sphere in the sheet the kth sphere. This sphere has mass m_k, equilibrium radius b_k, acoustic impedance[8] $z_k=m_k+4\pi\rho b_k^3$, and it is centered at position $\mathbf{r}_k$. If we take as our incoming plane wave the radiation pressure field of a very distant pulsating sphere (i.e., Eq. 3) then the pressure on the kth sphere will be

$$p(r_k)=-(i\omega\rho r_0^2V_0/r_k)\exp[i\omega(r_k/v-t)]. \quad (19)$$

If b_k is sufficiently small compared with the wavelength of incident radiation, the pressure can be taken to be constant over the entire sphere. The force on the kth sphere is then:

$$F_k=-(4\pi ib_k^2\omega\rho r_0^2V_0/r_k)\exp[i\omega(r_k/v-t)]. \quad (20)$$

Denoting by X the displacement of the spherical surface from its equilibrium position, we have

$$z_k\ddot{X}=-(4\pi ib_k^2\omega\rho r_0^2V_0/r_k)\exp[i\omega(r_k/v-t)]. \quad (21)$$

Since the kth sphere is simply reacting to pressure variations produced by a plane wave impinging on it, it is useful to rewrite Eq. 19 in the form

$$p(r_k)=p_0\exp(-i\omega t). \quad (22)$$

We now solve Eq. 21 for $\dot{X}$:

$$\dot{X}=-(4\pi b_k^2p_0/i\omega z_k)\exp(-i\omega t)\equiv\dot{X}_0\exp(-i\omega t). \quad (23)$$

This harmonic pulsation of the kth sphere will give rise to a radiation field:

$$p_k(r,t)=-(i\omega\rho b_k^2\dot{X}_0/r)\exp[i\omega(r/v-t)]. \quad (24)$$

We can now calculate the total field due to the pulsations of all the spheres in the sheet and thereby obtain the sought after reradiated pressure field. Since the width of the sheet is small, we may assume that, in response to the incident plane wave, all the spheres in it pulsate in phase with the kth sphere. The total field is then obtained from the integral:

$$p_{\text{rerad}}(x)=\int_0^\infty\frac{8\pi^2N\Delta Lb_k^4p_0}{z_k(x^2+h^2)^{\frac{1}{2}}}\exp(-i\omega t)\times\exp\left[\frac{i\omega}{v}(x^2+h^2)^{\frac{1}{2}}\right]hdh, \quad (25)$$

which is integrated over circular annuli of radii h and width dh.

Evaluating the integral in Eq. 25 gives[9]

$$p_{\text{rerad}}(x)=-(8\pi^2N\Delta L\rho b_k^4p_0v/i\omega z_k)\times\exp[-i\omega(t-x/v)]. \quad (26)$$

By comparing this result to Eq. 18 we finally obtain n:

$$n=1+8\pi^2N\rho v^2b_k^4/z_k\omega^2. \quad (27)$$

The authors wish to point out that this result for n does not agree with the index of refraction for a medium filled with hard spheres as calculated by Kock and Harvey.[10] Their result is independent of frequency and was obtained under the assumption that the sole effect of the hard spheres is to enhance the density of the medium by changing its total volume. The reasoning is via Rayleigh's principle. The present derivation takes detailed account of the shape and properties of a spherical radiator.

IV. CALCULATION OF RADIATION REACTION

Having obtained n, we shall evaluate the radiation reaction on a radiating source. We assume that the source is located at the origin and we will show that by summing one-half of the advanced pressure fields of all the receivers and evaluating that result at the source, we obtain the correct force of reaction. Let us begin by calculating the advanced field of just the kth receiver and evaluating it at the source. Recalling Eq. 24, we are tempted to write

$$p_{k\,\mathrm{adv}}(\text{source}) = -\frac{2\pi i\omega\rho^2 r_0^2 b_k^4 V_0}{r_k^2 z_k^2} \times \exp[i\omega(r_k/v - r_k/v - t)]. \quad (28)$$

This would be incorrect, however, because we have not taken into account the fact that the radiation from the source to the kth receiver travels not with velocity v, but with velocity v/n.[11] The correct result is, therefore,

$$p_{k\,\mathrm{adv}}(\text{source}) = -\frac{2\pi i\omega\rho^2 r_0^2 b_k^4 V_0}{r_k^2 z_k^2} \times \exp[i\omega(nr_k/v - r_k/v - t)]. \quad (29)$$

Using this expression for the pressure of a typical receiver, we obtain the total force on the emitter by integrating[9]

$$F_{\mathrm{reac}} = -\int_0^\infty 32\pi^3 i\omega\rho^2 r_0^4 N b_k^4 V_0 z_k^{-1} \exp(-i\omega t) \times \exp\left[\frac{8\pi^2 i\rho N v b_k^4 r_k}{\omega z_k}\right] dr_k \quad (30)$$

$$= \frac{4\pi\rho\omega^2 r_0^4 V_0}{v} \exp(-i\omega t),$$

which is in perfect agreement with the traditional result in Eq. 15. Note, also, that the final result is independent of the assumed properties of the absorber.

V. SUMMARY

One reason that Wheeler and Feynman wished to credit the radiation reaction to the absorber fields rather than to the source fields was to clear away the last obstacle to a complete action-at-a-distance formulation of electrodynamics. Such a description, they felt, would eliminate the troublesome infinities that arise when passing from classical to quantum electrodynamics. Since their paper was written, however, calculational schemes have been devised that remove these infinities and yet do not ask of the physicist that he give up the warm security of a field theory and purely retarded effects. As a consequence, the beautiful formulation of Wheeler and Feynman has gone largely untapped.

The present authors have as their ultimate objective to gain insight into certain aspects of acoustics and electrodynamics by exploiting the formal analogies between them. While we do not specifically contemplate transition to a "quantum acoustics," we feel it is an interesting and fruitful exercise to treat acoustics as an action-at-a-distance theory, wherein the medium plays as passive a role as does the vacuum in electrodynamics. Consonant with this view, we have demonstrated that one can postulate consistently that acoustic radiation is an interaction between a source and its absorbers and that the origin of the radiation reaction is the combined fields of all the receivers, acting in concert upon the source.

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[1] J. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

[2] P. A. M. Dirac, Proc. R. Soc. A **167**, 148 (1938).

[3] Note that the acoustical treatment is simpler than the original, as Wheeler and Feynman were denied the option of using monopole sources.

[4] S. N. Rschevkin, *A Course of Lectures on the Theory of Sound* (Pergamon Press, distributed by The Macmillan Co., New York, 1963), p. 79.

[5] Ref. 4, p. 87.

[6] This expression, along with a term that averages to zero over a full period is exactly what is usually obtained by calculating $4\pi r_0^2 p_{\mathrm{ret}}(r_0)$.

[7] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1966), Vol. 1, Sec. 31-1.

[8] I. Malecki, *Physical Foundations of Technical Acoustics* (Pergamon Press, Oxford, England, 1969), p. 175.

[9] Although the value of this integral and a similar one in Eq. 29 is mathematically indeterminate at the upper limit, there are good physical grounds for assigning them each the value zero at that limit. This is fully discussed in the Ref. 7 in Sec. 30-7 and in Ref. 1 on p. 162.

[10] W. E. Kock and F. K. Harvey, J. Acoust. Soc. Am. **21**, 471 (1949).

[11] Ref. 1, p. 161.