Overcounting Functions

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1. Introduction

Let $S = \sum_{k=1}^{\infty} f(k)$. The objective of this paper is to develop methods for evaluating sums of the form:

$$S^{+} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} f(g(k, j)), \tag{1.1}$$

where $g(k, j)$ ranges over the natural numbers. Because range($g(k, j)$) $\subseteq$ range(k), every value $l = g(k, j)$ corresponds to some index k in the sum S, so the effect of the double sum S^{+} in general is to count particular terms of S more than once.

For example, take

$$S = \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{1}{2}$$

If we now let $g(k, j) = k^j$, then

$$S^{+} = \sum_{j=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{2^{k^j}},$$

a sum that is very difficult to treat analytically. However, for $j = 1$, $g(k, j) = k$, so S^{+} includes every term of S and furthermore since k^j is a natural number, every term of S^{+} appears exactly once in S but possibly many times in S^{+}. Therefore S^{+} "overcounts" the terms of S. If we can determine the extent of overcounting, we can arrive at expressions for S^{+} that involve only a single sum.

We define the counting function $\mathrm{K}_g(k)$ as the number of times the term $f(k)$ of S is included in the double sum S^{+}:

$$S^{+} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(g(k,j)) = \sum_{k=1}^{\infty} \mathrm{K}_g(k) f(k) \tag{1.2}$$

It is useful to note that the double sum can be rewritten as the single sum on the right-hand side and that the form of $\mathrm{K}_g(k)$ depends only on $g(k,j)$ and not on $f(k)$.

It is sometimes more convenient to write (1.2) as

$$S^{+} = S + \sum_{k=1}^{\infty} (\mathrm{K}_g(k) - 1) f(k) = S + \sum_{k=1}^{\infty} \Omega_g(k) f(k) \tag{1.3}$$

where $\Omega_g(k) = \mathrm{K}_g(k) - 1$, which we shall call the overcounting function of *g*, is the number of times each term $f(k)$ of *S* is overcounted in the double sum S^{+}.

We may also rewrite *S* as the inner product of two infinite vectors *I* and *F*, where $I = \{1,1,1,\ldots\}$ is the identify vector and $F = \{f(1), f(2), \ldots, f(k), \ldots\}$. *I* is the characteristic vector of the index set of the sum *S*; $I_k = 1$ whenever *k* corresponds to a term in the sum s and is zero otherwise. In brief notation, $S = I \bullet F$. Then if $K = \{\mathrm{K}_g(1), \mathrm{K}_g(2), \ldots\}$, we have $S^{+} = K \bullet F$. *K* then is a vector indicating the multiplicity in S^{+} of each term of *S*.

2. Elementary Examples

One of the simplest examples of a counting function $\mathrm{K}_g(k)$ is obtained by taking $g(k,j) = k + j$. This results in each term $f(k)$ being counted $k-1$ times, once for each $1 \le j \le k-1$. So $\mathrm{K}_{k+j}(k) = k-1$. Therefore we may immediately write by inspection, for example, that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(k+j)^s} = \sum_{k=1}^{\infty} \frac{k-1}{k^s} = \zeta(s) - \zeta(s-1) \tag{2.1}$$

The ordinary generating function of the $\mathrm{K}_{k+j}(n)$ is

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x^{k+j} = \sum_{k=1}^{\infty} (k-1)x^k = \frac{x^2}{(x-1)^2} \tag{2.2}$$

We may also derive:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(k+j)^s (k+j-1)} = \sum_{k=1}^{\infty} \frac{k-1}{k^s (k-1)} = \zeta(s) \tag{2.3}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\zeta(k+j)-1 \;=\; \sum_{k=2}^{\infty}(k-1)(\zeta(k)-1) \;=\; \zeta(2) \tag{2.4}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2^{k+j}} \;=\; \sum_{k=1}^{\infty}\frac{k-1}{2^{k}} \;=\; 1 \tag{2.5}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{k+j}{2^{k+j}} \;=\; \sum_{k=1}^{\infty}\frac{k(k-1)}{2^{k}} \;=\; 4$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(k+j)2^{k+j}} \;=\; \sum_{k=1}^{\infty}\frac{k-1}{k2^{k}} \;=\; 1-\log 2 \tag{2.6}$$

Since $\sum_{k=1}^{\infty}\frac{1}{(k+j)2^{k+j}} = \frac{\Phi\left(\frac{1}{2},1,j+1\right)}{2^{j+1}}$, we immediately have

$$\sum_{j=1}^{\infty}\frac{\Phi\left(\frac{1}{2},1,j+1\right)}{2^{j+1}} = 1-\log 2$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\zeta(k+j)}{2^{k+j}} \;=\; \sum_{k=1}^{\infty}\frac{(k-1)\zeta(k)}{2^{k}} \;=\; \frac{\pi^2}{8} \tag{2.7}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(k+j)!} \;=\; \sum_{k=1}^{\infty}\frac{k-1}{k!} \;=\; 1 \tag{2.8}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{k+j}}{(k+j)!} \;=\; \sum_{k=1}^{\infty}\frac{(k-1)x^{k}}{k!} \;=\; (x-1)e^{x}+1 \tag{2.9}$$

A slightly more complicated counting function is associated with $g(k,j) = k\cdot j$. In this case, the term $f(l)$ is counted in S^{+} whenever there is a pair of indices (i,j) such that $i\cdot j = l$. The number of times this occurs is the same as the number of divisors $\sigma_0(l)$ of *l*. Therefore, $\mathrm{K}_{k\cdot j}(k) \;=\; \sigma_0(k)$ and we have:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(k\cdot j)^{s}} \;=\; \sum_{k=1}^{\infty}\frac{\sigma_0(k)}{k^{s}} \;=\; \zeta(s)^2 \tag{2.10}$$

The ordinary generating function of the $\mathrm{K}_{kj}(n)$ is

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x^{kj} = \sum_{k=1}^{\infty}\sigma_0(k)x^k = \sum_{k=1}^{\infty}\frac{x^k}{1-x^k} \tag{2.11}$$

Some additional examples:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{kj(kj+1)} = \sum_{k=1}^{\infty}\frac{\gamma+\psi\left(1+\frac{1}{k}\right)}{k} = \sum_{k=1}^{\infty}\frac{\sigma_0(k)}{k(k+1)} \tag{2.12}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\zeta(kj+1)-1 = \sum_{k=1}^{\infty}\sigma_0(k)(\zeta(k+1)-1) \tag{2.13}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{a^{k\cdot j}} = \sum_{k=1}^{\infty}\frac{\sigma_0(k)}{a^k} = \sum_{k=1}^{\infty}\frac{1}{a^k-1} \tag{2.14}$$

The latter follows from the identity

$$\sum_{k=1}^{\infty}\frac{1}{a^{k\cdot j}} = \frac{1}{a^k-1} \ . \tag{2.15}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(k\cdot j)!} = \sum_{k=1}^{\infty}\frac{\sigma_0(k)}{k!} \tag{2.16}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin(k\cdot j)}{(k\cdot j)} = \sum_{k=1}^{\infty}\frac{\sigma_0(k)\sin k}{k} \tag{2.17}$$

3. A Non-Trivial Example

Let $g(k,j)=k^j$. The number of times a particular summand $f(n)$ appears in S^+ for a particular value of n is the number of ways $W(n)$ that n can be expressed as an integer power of a natural number. We must turn our attention to finding an expression for $W(n)$.

Let $p_1^{n_1}p_2^{n_2}\ldots$ be the prime factorization of n . This can be the power of an integer greater than one only if $G(n) = \gcd(n_1,n_2,\ldots)$ is greater than one. If the n_i have no common divisor other than 1, then n cannot be of the form u^v, $u < n$, $v > 1$, since v would then divide each of the

n_i. Therefore, if $g(n) = 1$, then $W(n) = 1$, that is, n is a power of itself only. We now show that $W(n) = \sigma_0(G(n))$, where $\sigma_0(k)$ is the number of divisors of k.

Suppose b is a divisor of $G(n)$. Then $n = (p_1^{n_1/b} p_2^{n_2/b} \ldots)^b$, where each of the n_i / b is a natural number, so n is a power of a natural number. Now let c be a natural number that is not a divisor of $G(n)$. Then n cannot be the c^{th} power of any natural number, since in $(p_1^{n_1/c} p_2^{n_2/c} \ldots)^c$ at least one of the exponents n_i / c is not a natural number, so the corresponding factor $p_i^{n_i/c}$ is not a natural number and the product in parentheses is not a natural number.

Therefore,

$$\mathrm{K}_{k^j}(n) = \sigma_0(G(n)) \quad \text{and} \quad \Omega_{k^j}(n) = \sigma_0(G(n)) - 1 \tag{3.1}$$

$\Omega(k)$ measures the factor by which $S^+(a)$ overcounts the k^{th} term of S. Specifically,

$$S^+ = S + \sum_{k=2}^{\infty} \Omega_{k^j}(k) f(k) \tag{3.2}$$

Here are the first several non-zero values of $\Omega_{k^j}(n)$:

n	4	8	9	16	25	27	32	36	49	64	81	100	121	125
$\Omega_{k^j}(n)$	1	1	1	2	1	1	1	1	1	3	2	1	1	1

The ordinary generating function of the $\Omega_{k^j}(n)$ is given by:

$$\sum_{k=1}^{\infty} \Omega_{k^j}(k) x^k = \frac{x^2}{x-1} + \sum_{j=1}^{\infty} \sum_{k=2}^{\infty} x^{k^j} \tag{3.3}$$

Now we can examine particular cases. Let S $= \zeta(a) - 1$. That is, $f(k) = k^{-a}$.

$$S^+ = \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} \left(k^{-j}\right)^a = \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} k^{-ja} = \sum_{k=2}^{\infty} \frac{1}{k^a - 1} = \sum_{n=1}^{\infty} \left(\zeta(na) - 1\right)$$

Summing the identity

$$\frac{1}{k^s - 1} - \frac{1}{k^{2s} - k^s} = \frac{1}{k^s} \ ,$$

we have

$$S^{+}-S=\sum_{k=2}^{\infty}\frac{1}{k^{2a}-k^{a}}=\sum_{n=1}^{\infty}\left(\zeta\left((n+1)a\right)-1\right) \tag{3.4}$$

Therefore, from (3.2) it follows that

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k^{a}}=\sum_{k=2}^{\infty}\frac{1}{k^{2a}-k^{a}}=\sum_{n=1}^{\infty}\left(\zeta\left((n+1)a\right)-1\right) \tag{3.5}$$

For $a=1$, this yields the remarkable result that

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k} = \sum_{k=2}^{\infty}\frac{1}{k^{2}-k} = 1 \ , \tag{3.6}$$

giving the expansion

$$1 = \frac{1}{4}+\frac{1}{8}+\frac{1}{9}+\frac{2}{16}+\frac{1}{25}+\frac{1}{27}+\frac{1}{32}+\frac{1}{36}+\frac{1}{49}+\frac{3}{64}+\frac{2}{81}+\ldots$$

A simple way to understand this result is to ask for how many values of s does the term $1/n$ appear in the expansion of $\zeta(s)-1$? The answer is for each s such that n is a power of s, namely $\Omega_{k^j}(n)$. Therefore, we must have

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k}=\sum_{n=2}^{\infty}\left(\zeta(n)-1\right)=1$$

For other values of a, we obtain

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k^{2}} = \sum_{k=2}^{\infty}\frac{1}{k^{4}-k^{2}} = \frac{7}{4}-\zeta(2) \approx 0.10506593315177\ldots$$

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k^{3}} = \sum_{k=2}^{\infty}\frac{1}{k^{6}-k^{3}} = \zeta(3)-1+\sum_{k=2}^{\infty}\frac{1}{k^{3}-1} \approx 0.0196324919496727\ldots$$

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k^{4}} = \sum_{k=2}^{\infty}\frac{1}{k^{8}-k^{4}} = \frac{15}{8}-\frac{\pi}{4}\coth\pi-\zeta(4) \approx 0.0043397425545712\ldots$$

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k^{3/2}} = \sum_{k=2}^{\infty}\frac{1}{k^{3}-k^{3/2}} = \sum_{k=2}^{\infty}\zeta(3(k+1)/2)-1 \approx 0.283023507789527803\ldots$$

Now let $f(k) = (-1)^k k^{-a}$. This gives

$$S = \sum_{k=2}^{\infty} (-1)^k k^{-a} = 1 - \zeta(a)(1-2^{1-a})$$

$$S^+ = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty} (-1)^{k^j} k^{-ja} = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty} (-1)^k k^{-ja} .$$

The last equality follows from the fact that integer powers of even numbers are even and those of an odd number are odd, so k^j follows the parity of k. Thus,

$$S^+ = \sum_{k=2}^{\infty} (-1)^k \sum_{j=1}^{\infty} k^{-ja} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k^a - 1}.$$

For the special case $a = 1$ we have $S = 1 - \log 2$ and

$$S^+ = \sum_{k=2}^{\infty} \frac{(-1)^k}{k-1} = \log 2 \tag{3.7}$$

which gives

$$\sum_{k=2}^{\infty} (-1)^k \frac{\Omega(k)}{k} = 2\log 2 - 1 \approx 0.38629436111989... \tag{3.8}$$

For other values of a we have

$$\sum_{k=2}^{\infty} (-1)^k \frac{\Omega(k)}{k^2} = \zeta(2)(1-2^{-1}) - 1 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2-1} = \frac{\zeta(2)}{2} - \frac{3}{4}$$

$$\sum_{k=2}^{\infty} (-1)^k \frac{\Omega(k)}{k^3} = \zeta(3)(1-2^{-2}) - 1 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^3-1}$$

$$\sum_{k=2}^{\infty} (-1)^k \frac{\Omega(k)}{k^4} = \zeta(4)(1-2^{-3}) - 1 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^4-1} = \frac{7\zeta(4)}{8} - \frac{7}{8} - \frac{\pi}{4\sinh \pi}$$

With $S = \sum_{k=2}^{\infty} \frac{1}{k!} = e - 2$, we obtain

$$S^+ = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(k^j)!} = e - 2 + \sum_{k=2}^{\infty}\frac{\Omega(k)}{k!} \quad .$$

For $S = \sum_{k=2}^{\infty}\frac{1}{a^k} = \frac{1}{a(a-1)}$, we have

$$S^+ = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{a^{k^j}} = \frac{1}{a(a-1)} + \sum_{k=2}^{\infty}\frac{\Omega(k)}{a^k} \quad .$$

In particular,

$$\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^{k^j}} = \frac{1}{2} + \sum_{k=2}^{\infty}\frac{\Omega(k)}{2^k} \quad .$$

From $S = \sum_{k=2}^{\infty}\frac{1}{k(k+1)}$ we derive

$$\sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{k^j(k^j+1)} = \sum_{k=2}^{\infty}\frac{\Omega(k)}{k(k+1)} \approx 0.088947552613725...$$

$S(a) = \sum_{k=2}^{\infty}\frac{k}{2^k} = \frac{3}{2}$ gives

$$\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{k^j}{2^{k^j}} = \frac{3}{2} + \sum_{k=2}^{\infty}\frac{\Omega_{k^j}(k)k}{2^k} \approx 1.799317360448272406...$$

$$\sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{\log\left(k^j\right)}{\left(k^j\right)^2} = \sum_{k=2}^{\infty}\frac{\log k}{k^2(1-k^{-2})^2} = \sum_{k=2}^{\infty}\frac{\Omega_{k^j}(k)\log k}{k^2} \tag{3.9}$$

$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k(k+1)} = \sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{k^j\left(k^j+1\right)} = \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}(-1)^k\left(\zeta(mk+m)-1\right)$$
$$\sum_{k=2}^{\infty}\frac{\Omega(k)}{k(k-1)} = \sum_{k=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{k^j\left(k^j-1\right)} = \sum_{k=1}^{\infty}\sum_{m=2}^{\infty}\left(\zeta(mk+m)-1\right)$$

The right-hand equalities follow from the relations

$$\sum_{k=2}^{\infty}\frac{1}{k^j\left(k^j+1\right)} = \sum_{n=2}^{\infty}(-1)^{n+1}\left(\zeta(jn+j)-1\right)$$

and

$$\sum_{k=2}^{\infty}\frac{1}{k^j\left(k^j-1\right)}=\sum_{n=2}^{\infty}\left(\zeta(jn+j)-1\right)$$

Note that for p prime, $\Omega(p^k) = \sigma_0(k)-1$.

$$\sum_{k=2}^{\infty}\frac{\Omega(p^k)}{p^k} = \sum_{k=1}^{\infty}\sum_{j=2}^{\infty}\frac{1}{p^{jk}}$$

$$\sum_{k=2}^{\infty}\frac{\Omega(p)}{p^k} = \sum_{k=1}^{\infty}\sum_{j=2}^{\infty}\frac{1}{p^{j^k}}$$

We may now consider values of $\Omega(k^a)$. For example, it is possible to compute

$$T(2) = \sum_{k=2}^{\infty}\frac{\Omega(k^2)}{k^2}$$

via the following argument. The overcounting function $\Omega(k^2)$ enumerates the number of appearances of k^{-2} in the semi-infinite matrix

$$R=\left[r_{i,j}\right] \qquad ; r_{i,j}=\frac{1}{(j+1)^{i+1}}$$

The sum of row i of R is given by:

$$\sum_{j=1}^{\infty}\frac{1}{(j+1)^{i+1}} = \zeta(i+1)-1$$

A term of the form k^{-2} appears in every odd row of R. Such a term also appears in an even row $2i$ of R only when it is of the form $(k^{-2})^{2i+1}$. Therefore,

$$\sum_{k=2}^{\infty}\frac{\Omega(k^2)}{k^2}=\sum_{k=1}^{\infty}\left(\zeta(2k)-1\right)+\sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{\left(k^2\right)^{2i+1}}$$

However,

$$\sum_{k=1}^{\infty}\left(\zeta(2k)-1\right)=\frac{3}{4}$$

and

$$\sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{\left(k^2\right)^{2i+1}}=\sum_{k=2}^{\infty}\sum_{i=1}^{\infty}\frac{1}{\left(k^2\right)^{2i+1}}=\sum_{k=1}^{\infty}\left(\zeta(4k+2)-1\right)=\sum_{k=2}^{\infty}\frac{1}{k^6-k^2}=\frac{7}{8}-\zeta(2)+\frac{\pi}{4}\coth\pi$$

Therefore,

$$T(2)=\sum_{k=2}^{\infty}\frac{\Omega(k^2)}{k^2}=\frac{13}{8}-\zeta(2)+\frac{\pi}{4}\coth\pi\approx 0.768402956886064\ldots .$$

To compute T(3), note that all the terms of row i of R are included in the summation where $(i+1)\equiv 0\pmod 3$ and only those terms in the other rows are included that are of the form $\left(k^3\right)^{-(i+1)}$. This is true because n^{i+1} cannot be a perfect cube if $(i+1)\equiv 1$ or $2\pmod 3$ unless n is a perfect cube. Therefore,

$$\begin{aligned}T(3)=\sum_{k=2}^{\infty}\frac{\Omega(k^3)}{k^3} &= \sum_{k=1}^{\infty}\left(\zeta(3k)-1\right)+\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}\frac{1}{\left(k^3\right)^{3i+2}}+\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}\frac{1}{\left(k^3\right)^{3i+4}}\\ &= \sum_{k=2}^{\infty}\frac{1}{k^3-1}+\sum_{k=2}^{\infty}\frac{1}{k^6-k^{-3}}+\sum_{k=2}^{\infty}\frac{1}{k^{12}-k^3}\\ &=\sum_{k=1}^{\infty}\left(\zeta(3k)-1\right)+\sum_{k=1}^{\infty}\left(\zeta(9k-3)-1\right)+\sum_{k=1}^{\infty}\left(\zeta(9k+3)-1\right)\\ &\approx 0.239309669474300\ldots\end{aligned}$$

Likewise for $T(4)$, except that when $(i+1)\equiv 2\pmod 4$, a term is a perfect fourth power when it is an even power of a term of the form k^2. Hence,

$$\begin{aligned}T(4)=\sum_{k=2}^{\infty}\frac{\Omega(k^4)}{k^4} &= \sum_{k=1}^{\infty}\left(\zeta(4k)-1\right)+\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}\frac{1}{\left(k^2\right)^{4i+2}}+\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}\frac{1}{\left(k^4\right)^{4i+3}}+\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}\frac{1}{\left(k^4\right)^{4i+5}}\\ &= \sum_{k=2}^{\infty}\frac{1}{k^4-1}+\sum_{k=2}^{\infty}\frac{1}{k^4-k^{-4}}+\sum_{k=2}^{\infty}\frac{1}{k^{12}-k^{-4}}+\sum_{k=2}^{\infty}\frac{1}{k^{20}-k^4}\\ &=\sum_{k=1}^{\infty}\left(\zeta(4k)-1\right)+\sum_{k=1}^{\infty}\left(\zeta(8k-4)-1\right)+\sum_{k=1}^{\infty}\left(\zeta(16k-4)-1\right)+\sum_{k=1}^{\infty}\left(\zeta(16k+4)-1\right)\end{aligned}$$

$\approx 0.169480298487417...$

In general,

$$T(a)=\sum_{k=2}^{\infty}\frac{\Omega(k^a)}{k^a}=\sum_{k=1}^{\infty}\left(\zeta(ak)-1\right)+\sum_{m=2}^{a-1}\sum_{i=0}^{\infty}\sum_{k=2}^{\infty}\frac{1}{\left(k^a\right)^{ai+m}}+\sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{\left(k^a\right)^{ai+1}}\quad .$$

4. Some LCM Examples

Let $g(k,j)=\mathrm{LCM}(k,j)$, the least common multiple of k and j. The number of times $f(n)$ appears in S^+ for a particular value of n is the number $M(n)$ of ordered pairs of natural numbers whose LCM is n. It is shown in a separate paper that

$$M(n)=\prod_{i=1}^{\nu(n)}(2e_i+1)\;=\;\sigma_0(n^2),\qquad n>1 \tag{4.1}$$

where $\nu(n)$ is the number of distinct prime factors of n and e_i is the exponent of the i^{th} prime in the prime factorization of n. Conventionally, we take $M(1) = 1$.

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{\left(\mathrm{LCM}(k,j)\right)^a}=\sum_{k=1}^{\infty}\frac{M(k)}{k^a}=\sum_{k=1}^{\infty}\frac{\sigma_0(k^2)}{k^a}=\frac{\zeta^3(a)}{\zeta(2a)} \tag{4.2}$$

The last equality on the right is from Titchmarsh 1.2.9.

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^{\mathrm{LCM}(k,j)}}=\sum_{k=1}^{\infty}\frac{M(k)}{2^k}=1+\sum_{k=1}^{\infty}\frac{2^{\nu(k)}}{2^k-1} \tag{4.3}$$

The quantity $2^{\nu(k)}$ is the number of quadratfrei divisors of k, that is, divisors consisting of a product of distinct primes. (See Hardy & Wright 17.8.)

5. Sums of Squares

Let $g(k,j)=k^2+j^2$. The number of ways $q(n)$ that n can be expressed as the sum of two squares of natural numbers has the generating function

$$Q(x) = \sum_{k=1}^{\infty} q(k) x^k = \left(\sum_{m=1}^{\infty} x^{m^2} \right)^2 \tag{5.1}$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{a^{k^2+j^2}} = \left(\sum_{m=1}^{\infty} a^{-m^2} \right)^2 = \sum_{k=1}^{\infty} \frac{q(k)}{a^k} \tag{5.2}$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\left(k^2 + j^2\right)^c} = \sum_{k=1}^{\infty} \frac{q(k)}{k^a} \tag{5.3}$$

6. Multiple Products

We may transform multiple products in a manner similar to multiple sums, except that the counting function $\mathrm{K}_g(k)$ then appears as an exponent rather than as a factor.

In general,

$$\prod_{j=1}^{\infty} \prod_{k=1}^{\infty} a_{g(k,j)} = \prod_{k=1}^{\infty} (a_k)^{\mathrm{K}_g(k)} \tag{6.1}$$

For example,

$$\prod_{j=1}^{\infty} \prod_{k=1}^{\infty} \left(1 + \frac{1}{(k+j)^2} \right) = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2} \right)^{k-1} \tag{6.2}$$

Unfortunately, (6.2) does not converge. This may be seen by rewriting the product as

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2} \right)^{k-1} = \prod_{k=1}^{\infty} \left(1 + \sum_{j=1}^{k-1} \binom{k-1}{j} \frac{1}{k^{2j}} \right) \tag{6.3}$$

The right-hand side of (6.3) converges if and only if the following double sum converges:

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \binom{k-1}{j} \frac{1}{k^{2j}} = \sum_{k=1}^{\infty} \left(\frac{k^2 \left(1 + \frac{1}{k^2} \right)^k}{k^2 + 1} - 1 \right) \tag{6.4}$$

From the relation

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{(k+j)^2}\right)=\frac{\Gamma^2(j+1)}{\Gamma(j+1-i)\Gamma(j+1+i)}=\frac{\sinh\pi}{\pi}\frac{(j!)^2}{\prod_{k=1}^{j}k^2+1} \tag{6.3}$$

The quantities $c_j = \dfrac{(j!)^2}{\prod_{k=1}^{j}k^2+1}$ are of some interest. They are all rational, and in the limit as j grows large, approach the limit $\pi/\sinh\pi$. In general, we have

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{k^a+1}\right)=\lim_{j\to\infty}\frac{(j!)^a}{\prod_{k=1}^{j}k^a+1} \tag{6.4}$$

$$\prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(1-\frac{1}{2^{k+j}}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)^{k-1} \tag{6.5}$$

$$\prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(1-\frac{1}{2^{kj}}\right)=\prod_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)^{\sigma_0(k)} \tag{6.6}$$

7. Partition Products

The idea of overcounting also has application to infinite products. Consider Euler's product expansion of the zeta function:

$$\zeta(s) = \prod_{p=2}^{\infty}\frac{1}{1-p^{-s}} = \sum_{k=1}^{\infty}\frac{1}{k^s}, \tag{7.1}$$

where the sum is taken over all primes p. The previous investigation leads us to inquire what the result would be if the product were over all natural numbers greater than two instead of just over the primes.

Let $z(n)$ be the total number of factorizations of n into products of factors, not necessarily prime, in which factorizations that differ only in the order of their factors are not treated as distinct.

For n prime, $z(n) = 1$; that is, the only factorization is $1 \cdot n$. If n is a product of k distinct prime factors, then we have

$$z(n) = b_k = \sum_{m=1}^{k} S_2(k,m) \ , \tag{7.2}$$

where $S_2(k,m)$ is a Stirling number of the second kind and b_k is the k^{th} Bell number, which gives the total number of partitions of a set of k distinct elements. The first several values of b_k are:

k	1	2	3	4	5	6	7	8	9	10
b_k	1	2	5	15	52	203	877	4140	21147	115975

If n is the k th power of a prime, then any partition of the number k corresponds to a factorization of n. For example, the partition 7 = 3 + 2 + 2 corresponds to the product $p^3 p^2 p^2$.

$z(n) \ = \ p(k)$, the number of unordered partitions of the integer k.

k	1	2	3	4	5	6	7	8	9	10	11	12
$p(k)$	1	2	3	5	7	11	15	22	30	42	56	77

The number of factorizations of a natural number depends only on the specification of the set of exponents of its prime factors. Thus

$$36 \ = \ 2^2 3^3 \qquad \text{and} \qquad 3025 \ = \ 5^2 11^2$$

both have the same number (9) of factorizations.

Now, $z(k)$ is a type of overcounting function because it measures the number of times each term of the form k^{-s} appears when the product (10.1) is extended over natural numbers ≥ 2 instead of just over primes p:

$$\prod_{n=2}^{\infty} \frac{1}{1-n^{-s}} = \sum_{k=1}^{\infty} \frac{z(k)}{k^s} \ . \tag{7.3}$$

Further Partition Results

$$\left(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+...\right)\left(1+\frac{1}{a^2}+\frac{1}{a^4}+\frac{1}{a^6}+...\right)\left(1+\frac{1}{a^3}+\frac{1}{a^6}+\frac{1}{a^9}+...\right)...$$

$$= \ \prod_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{a^{jk}} \ = \ \prod_{k=1}^{\infty}\frac{1}{1-a^{-k}} \ = \ \frac{a+1}{a}+\sum_{k=1}^{\infty}\frac{p(k)}{a^k}$$

This follows from Euler's generating function $P(x)$ for the partition numbers $p(n)$:

$$P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{n=1}^{\infty} p(n)x^n$$

If $q(n)$ is the number of partitions of n into distinct parts, then

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{a^k}\right)=1+\sum_{n=1}^{\infty}\frac{q(k)}{a^k}=\prod_{k=1}^{\infty}\frac{a^{2k-1}}{a^{2k-1}-1}$$

8. Remarks

We observe that the idea of computing a sum by including too many summands and then subtracting the excess is a familiar one in combinatorics, notably in the Principle of Inclusion-Exclusion.